S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 6, стр. 510–513 (2009) УДК 514.8, 515.12
Краткие сообщения MSC 28A80

A SELF-SIMILAR CONTINUUM WHICH IS NOT THE ATTRACTOR OF ANY ZIPPER

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Abstract. The article contains a construction of a self-similar dendrite which is not the attractor of any self-similar zipper.

Keywords: self-similar sets, fractal, self-similar dendrites, zippers.

1. Introduction

Let $\mathcal{S}$ be a system $\{S_1, ..., S_m\}$ of injective contraction mappings of a complete metric space (X, d) to itself and let K be it's *invariant set*, that is a non-empty compact set K satisfying $K = \bigcup_{i=1}^{m} S_i(K)$. The set K is also called the *attractor of the system* $\mathcal{S}$. There is a natural construction allowing to obtain the systems $\mathcal{S}$ with an arcwise connected invariant set. This construction called a self-similar zipper goes back to the works of Thurston [5] and Astala [2] and was analyzed in detail by Aseev, Kravtchenko and Tetenov in [6]. Namely,

Definition 1. *A system $\mathcal{S} = \{S_1, ..., S_m\}$ of injective contraction maps of complete metric space X to itself is called a* zipper *with vertices $(z_0, ..., z_m)$ and signature $\vec{\varepsilon} = (\varepsilon_1, ..., \varepsilon_m) \in \{0, 1\}^m$ if for any $j = 1, ..., m$ $S_j(z_0) = z_{j-1+\varepsilon_j}$ and $S_j(z_m) = z_{j-\varepsilon_j}$.*

If the maps S_i are similarities (or affine maps) the zipper is called self-similar (correspondingly self-affine).

We shall call the points z_0 and z_m *the initial* and *the final* point of the zipper respectively.

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The work is supported by RFFI (grant 09-01-90202).

Communicated by A.D. Mednykh November, 30, 2009, published December, 4, 2009.

The simplest example of a self-similar zipper may be obtained if we take a partition P, $0 = x_0 < x_1 < \ldots < x_m = 1$ of the segment $I = [0, 1]$ into m parts and put $T_i = x_{i-1+\varepsilon_i}(1-t) + x_{i-\varepsilon_i}t$. This zipper $\{T_1, \ldots, T_m\}$ will be denoted by $\mathcal{S}_{P,\vec{\varepsilon}}$.

Theorem 2.(see [6]). *For any zipper $\mathcal{S} = \{S_1, ..., S_m\}$ with vertices $\{z_0, \ldots, z_m\}$ and signature $\vec{\varepsilon}$ in a complete metric space (X, d) and for any partition $0 = x_0 < x_1 < \ldots < x_m = 1$ of the segment $I = [0, 1]$ into m parts there exists an unique map $\gamma : I \to K(\mathcal{S})$ such that for each $i = 1, ..., m$, $\gamma(x_i) = z_i$ and $S_i \cdot \gamma = \gamma \cdot T_i$ (where $T_i \in \mathcal{S}_{P,\vec{\varepsilon}}$), the map γ being Hölder continuous.*

The mapping γ in the Theorem 2 is called a *linear parametrization* of the zipper $\mathcal{S}$. Thus, the attractor K of any zipper $\mathcal{S}$ is an arcwise connected set, whereas the linear parametrization γ may be viewed as a self-similar Peano curve, filling the continuum K.

Some Peano curves. The attractor K of a self-similar zipper $\mathcal{S}$ with vertices $(0, 0)$, $(1/4, \sqrt{3}/4)$, $(3/4, \sqrt{3}/4)$, $(1, 0)$ and signature $(1, 0, 1)$ is the Sierpinsky gasket.

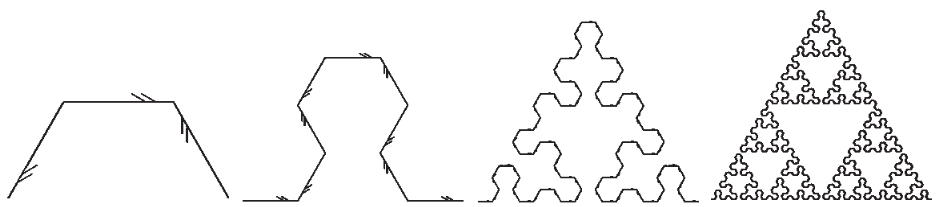

FIGURE 1. Iterations 1,2,4,∞ for Serpinsky gasket.

A self-similar zipper with vertices $(0, 0)$, $(0, 1/2)$, $(1/2, 1/2)$, $(1, 1/2)$, $(1, 0)$ and signature (1,0,0,1) produces a self-similar Peano curve for the square $[0, 1] \times [0, 1]$

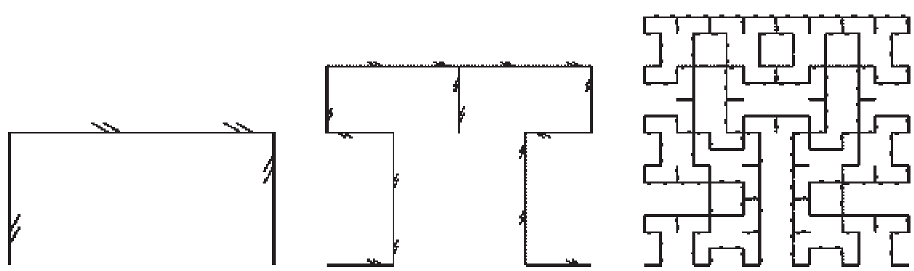

FIGURE 2. Iterations 1,2,4 for square-filling Peano curve.

2. THE MAIN EXAMPLE.

The following example shows the existence of a self-similar continuum which cannot be represented as the attractor of a self-similar zipper.

Let $\mathcal{S}$ be a system of contraction similarities g_k in $\mathbb{R}^2$ where $S_2(\vec{x}) = \vec{x}/2 + (2, 0)$, and $S_k(\vec{x}) = \vec{x}/4 + \vec{a}_k$ where $\vec{a}_k$ run through the set $\{$ $(0, 0)$, $(3, 0)$, $(1, 2h)$, $(3/2, 3h)\}$, $h = \sqrt{3}/2$ for $k = 1, 3, 4, 5$. Let K be the invariant set of the system $\mathcal{S}$ and T − the Hutchinson operator of the system $\mathcal{S}$ defined by $T(A) = \bigcup\limits_{j=1}^{5} S_j(A)$.

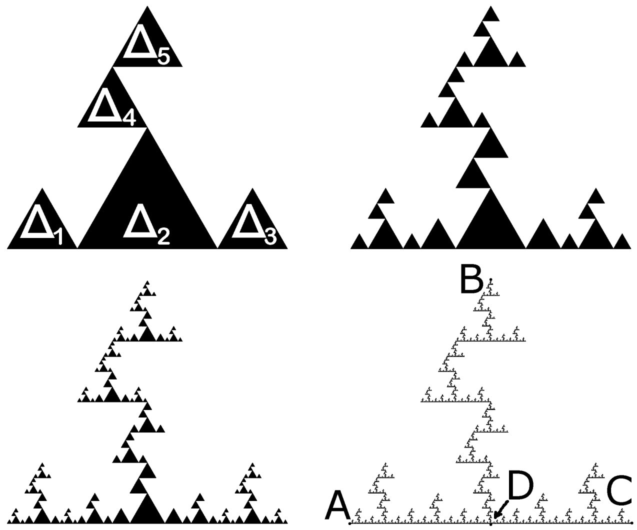

FIGURE 3. Iterations 1,2,4,∞ for the example.

We shall use the following notation: By Δ we denote the triangle with vertices $A = (0,0)$, $B = (2, 2\sqrt{3})$ and $C = (4,0)$, that is the convex hull of the points A, B and C. The point $(2,0)$ is denoted by D. Since for each S_i, $S_i(\Delta) \subset \Delta$, the invariant set K of the system $\mathcal{S}$ lies in Δ. For a multiindex $\mathbf{i} = i_1...i_k$ we denote $S_\mathbf{i} = S_{i_1}...S_{i_k}$, $\Delta_\mathbf{i} = S_\mathbf{i}(\Delta)$, $K_\mathbf{i} = S_\mathbf{i}(K)$, $A_\mathbf{i} = S_\mathbf{i}(A)$, etc.

1. *The set K is a dendrite.* The way the system $\mathcal{S}$ is defined (see [3, Thm.1.6.2]) guarantees the arcwise connectedness of K. Since for each n the set $T^n(\Delta)$ is simply-connected, the set K is a continuum, which contains no cycles, or a dendrite [4, Ch.6, §52]. Each point of K has the order 2 or 3. If a point x has the order 3, it is an image $S_\mathbf{i}(D)$ of the point D for some multiindex $\mathbf{i}$. Any path in K connecting a point $\xi \in J$ with a point $\eta \in \Delta_\mathbf{i}, \mathbf{i} = 4, 5, 24, 25, 224, 225, ..$, passes through the point D.

2. *Each non-degenerate segment J, contained in K is parallel to x axis and is contained in some maximal segment in K which has the length 4^{1-n}.*

Consider a non-degenerate linear segment $J \subset K$. There is such multiindex $\mathbf{i}$, that J meets the boundary of $S_\mathbf{i}(\Delta)$ in two different points which lie on different sides of $S_\mathbf{i}(\Delta)$ and do not lie in the same subcopy of $K_\mathbf{i}$. Then $J' = g_\mathbf{i}^{-1}(J \cap K_\mathbf{i})$ is a segment in K with endpoints lying on different sides of D which is not contained in neither of subcopies $K_1, ..., K_5$ of K. Then $J' = [0,4]$. Since a part of J is a base of some triangle $S_\mathbf{i}(\Delta)$, the length of the maximal segment in K containing J is 4^{1-n} where $n \le |\mathbf{i}|$.

3. *Any injective affine mapping f of K to itself is one of the similarities $S_\mathbf{i} = S_{i_1} \cdot ... \cdot S_{i_k}$.* Since f maps $[0,4]$ to some $J \subset S_\mathbf{i}([0,4])$ for some $\mathbf{i}$, it is of the form $f(x,y) = (ax + b_1 y + c_1, b_2 y + c_2)$, with positive b_2. Choosing appropriate composition $S_\mathbf{i}^{-1} \cdot f \cdot S_\mathbf{j}(K)$ we obtain a map of K to itself sending $[0,4]$ to some subset of $[0,4]$. Therefore we may suppose that $f(x,y) = (ax + b_1 y + c_1, b_2 y)$, and that the image $f(\Delta)$ is contained in Δ and is not contained in any $\Delta_i, i = 1, ..., 5$.

If $f(B) \in \Delta_\mathbf{i}, \mathbf{i} = 4, 5, 24, 25$, then $f(D) = D$ and $c_1 = 2 - a$.

If $f(B) \in \Delta_i, i = 4, 5$, then $1/2 \le b_2 \le 1$. In this case $y-$coordinates of the points $f(B_1), f(B_3)$ are greater than $\sqrt{3}/4$, so they are contained in Δ_1 and Δ_3, therefore the map f either keeps the points D_1, D_3 invariant, or transposes them. In each case $|a| = 1$ and $f(\{A, C\}) = \{A, C\}$. If in this case $f(B) \neq B$, then $f(A_4)$ cannot be contained in $T(\Delta)$. The same argument shows that if $f(B) = B$, then $f(A) \neq C$. Therefore $f = \mathrm{Id}$.

If $f(B) \in \Delta_i, i = 24, 25$, and $a > 1/2$ then the points $f(B_1), f(B_3)$ are again contained in Δ_1 and Δ_3, therefore the map f either keeps the points D_1, D_3 invariant, or transposes them, so $|a| = 1$ and $f(\{A, C\}) = \{A, C\}$. Considering the intersections of the line segments $[A, f(B)]$ and $[f(B), C]$ with the boundary of $T(\Delta)$ and $T^2(\Delta)$ we see that either $f(A_4)$ or $f(C_5)$ are not contained in $T^2(\Delta)$, which is impossible.

Therefore, either $a \le 1/2$ or $f = \mathrm{Id}$. The first means that $f(\Delta) \subset \Delta_2$, which contradicts the original assumption, so $f = \mathrm{Id}$.

4. *The set K cannot be an attractor of a zipper.* Let $\Sigma = \{\varphi_1, ..., \varphi_m\}$ be a zipper whose invariant set is K. Let x_0, x_1 be the initial and final points of the zipper Σ. Let γ be a path in K connecting x_0 and x_1. Since for every $i = 1, ..., m$ the map φ_i is equal to some $S_{\mathbf{j}}$, the sets $\varphi_i(K)$ are the subcopies of K, therefore for each i at least one the images $\varphi_i(x_0), \varphi_i(x_1)$ is contained in the intersection of $\varphi_i(K)$ with adjacent copies of K. Consider the path $\tilde{\gamma} = T_\Sigma(\gamma) = \bigcup\limits_{i=1}^{m} \varphi_i(\gamma)$. It starts from the point x_0, ends at x_1 and passes through all copies K_j of K. Each of the points $C_1 = A_2$, $C_2 = A_3$, $B_2 = C_4$ and $B_4 = A_5$ splits K to two components, therefore is contained in $\tilde{\gamma}$ and is a common point for the copies $\varphi_i(\gamma)$, $\varphi_{i+1}(\gamma)$ for some i. Therefore one of the points x_0, x_1 must be A, one of the points x_0, x_1 must be B, and one of the points x_0, x_1 must be C, which is impossible.

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