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ON SELFSIMILAR JORDAN CURVES ON THE PLANE

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Abstract: We study the attractors of a finite system of planar contraction similarities S_j $(j = 1, \dots, n)$ satisfying the coupling condition: for a set $\{x_0, \dots, x_n\}$ of points and a binary vector $(s_1, \dots, s_n)$, called the signature, the mapping S_j takes the pair $\{x_0, x_n\}$ either into the pair $\{x_{j-1}, x_j\}$ (if $s_j = 0$) or into the pair $\{x_j, x_{j-1}\}$ (if $s_j = 1$). We describe the situations in which the Jordan property of such attractor implies that the attractor has bounded turning, i.e., is a quasiconformal image of an interval of the real axis.

Keywords: attractor, selfsimilar fractal, open set condition, curve with bounded turning, quasiconformal mapping, quasiarc, Hausdorff measure, Hausdorff dimension, similarity dimension

In this article we study the attractors of a finite system S_j $(j = 1, \dots, n)$ of contraction similarities of a complete metric space $\mathscr{X}$ which satisfy the following *coupling condition*: for a set $\{x_0, \dots, x_n\}$ of points in $\mathscr{X}$, called *vertices,* and a binary vector $(s_1, \dots, s_n)$, called the *signature*, the mapping S_j $(i = 1, \dots, n)$ takes the pair $\{x_0, x_n\}$ either into the pair $\{x_{j-1}, x_j\}$ (if $s_j = 0$) or into the pair $\{x_j, x_{j-1}\}$ (if $s_j = 1$). We call such systems of similarities *zippers.* The basic properties of attractors of systems of contractions in a complete metric space, including existence and uniqueness theorems for an attractor, can be found in Hutchinson's fundamental article [1] or Crownover's monograph [2]. Some properties of zippers are described in § 3.5 of [1]. As Example 1.4 shows, the attractor of a zipper may fail to be a Jordan arc in general. In this article we do not dwell upon sufficient conditions on a zipper to possess the Jordan property and study the regularity properties of its attractor that follow only from the Jordan property. In § 4 we describe the situation on the plane in which the attractor of a Jordan zipper is an arc with bounded turning, i.e., a quasiconformal image of an interval of the real axis. In particular, these situations are

(1) the case of an alternating signature: $s_{j-1} + s_j = 1$ for all $j = 1, \dots, n$;

(2) the case in which $s_1 + s_n \geq 1$;

(3) the case of rational commensurability of the numbers $\mathrm{Log}(|x_1 - x_0|/|x_n - x_0|)$ and $\mathrm{Log}(|x_n - x_{n-1}|/|x_n - x_0|)$.

In § 2 we construct an example of a Jordan zipper on the plane whose attractor is not an arc with bounded turning. In Theorem 2.2 we establish that if the attractor of a Jordan zipper has bounded turning then it is a set of finite nonzero α-dimensional Hausdorff measure, where α is the similarity dimension of the system $\mathbf{S}$ which coincides in this case with the Hausdorff dimension of the attractor. Note that, while deriving Lemma 1.1 on a continuous structure parametrization of the attractor of a zipper, we cannot follow the proof of a similar assertion of [1, Theorem (3), p. 731] owing to the presence of an implicit requirement in [1] that the metric space under consideration is connected.

1. Attractors, zippers, and their parametrizations. A mapping $S : (\mathscr{X}_1, \rho_1) \to (\mathscr{X}_2, \rho_2)$ between metric spaces is a *contraction* if

$$\mathrm{Lip}(S) := \sup\{\rho_2(S(x), S(y))/\rho_1(x, y) : x, y \in \mathscr{X}_1,\ x \neq y\} < 1.$$

Given a natural $n \in \mathbb{N}$, put $I = \{1, \dots, n\}$, $I^k = \prod_{j=1}^{k} I$, where $k \in \mathbb{N}$, $I^* = \bigcup_{k=1}^{\infty} I^k$, and $I^\infty = \prod_{j=1}^{\infty} I$. The elements of I^k and I^* are written down as words of finite length in the alphabet I. The set I^* of multi-indices is the free semigroup generated by the elements of I with the concatenation operation, i.e.,

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simply joining one word to the other. The elements of I^∞ are written as words of infinite length in the alphabet I.

For a collection $\mathbf{S} = \{S_1, \dots, S_n\}$ of contractions of a complete metric space $(\mathscr{X}, \rho)$ into itself and an arbitrary $i = i_1 \dots i_k \in I^k$ we use the abbreviation $S_i = S_{i_1 \dots i_k} = S_{i_1} \circ \dots \circ S_{i_k}$. Given arbitrary $i = i_1 i_2 \dots \in I^\infty$ and $k \in \mathbb{N}$, we put $i|_k = i_1 \dots i_k \in I^k$. In the space $\mathrm{Comp}(\mathscr{X})$ of all nonempty compact subsets $A \subset \mathscr{X}$ with the metric given by the *Hausdorff distance* [3, Chapter 2, § 21, p. 223], the system $\mathbf{S}$ defines the *Hutchinson operator* $\Phi : \mathrm{Comp}(\mathscr{X}) \to \mathrm{Comp}(\mathscr{X})$ that takes a nonempty compact set $A \subset \mathscr{X}$ into the nonempty compact set $\Phi(A) = \bigcup_{k=1}^n S_k(A)$ (see [1] or [2, (4.1), p. 99]). The fixed point of Φ, i.e., the nonempty compact set $K(\mathbf{S}) \in \mathrm{Comp}(\mathscr{X})$ satisfying the equality $K(\mathbf{S}) = \Phi(K(\mathbf{S}))$, is called the *attractor* (or *invariant set*) of $\mathbf{S}$. The formula $\pi(i) = \lim_{k\to\infty} S_{i|_k}(x)$ correctly defines a continuous mapping $\pi : I^\infty \to K(\mathbf{S})$ independent of the choice of $x \in \mathscr{X}$ (where I^∞ is equipped with the Tychonoff product of discrete topologies) which is called the *index parametrization* of the attractor $K(\mathbf{S})$ of $\mathbf{S}$ (see [1, (vii), p. 725]). For $i \in I^k$, the subsets $K_i(\mathbf{S}) := S_i(K(\mathbf{S}))$ of the attractor $K(\mathbf{S})$ are referred to as *copies of rank* k. In Thurstone's terminology (see [4]), a system $\mathbf{S} = \{S_1, \dots, S_n\}$ of contractions of a complete metric space $(\mathscr{M}, \rho)$ into itself which satisfies the condition "there is a collection of points $\{x_0, \dots, x_n\} \subset \mathscr{M}$ and a vector $(s_1, \dots, s_n) \in \{0, 1\}^n$ for which $S_j(x_0) = x_{j-1+s_j}$ and $S_j(x_n) = x_{j-s_j}$ for all $j \in I$" is called a *zipper* with *vertices* $\{x_0, \dots, x_n\}$ and *signature* $(s_1, \dots, s_n)$. The following lemma generalizes Hutchinson's assertions [1, § 3.5, pp. 730–731] for zippers with signature $s_j = 0$ for all $j \in I$:

Lemma 1.1. *For every zipper $\mathbf{S} = \{S_1, \dots, S_n\}$ with vertices $\{x_0, \dots, x_n\}$ and signature $(s_1, \dots, s_n)$ in a complete metric space $(\mathscr{M}, \rho)$ and for every collection of points $0 = t_0 < t_1 < \dots < t_n = 1$ on the interval $J = [0, 1] \subset \mathbb{R}^1$, there is a unique mapping $\gamma : J \to K(\mathbf{S})$ such that $\gamma(t_i) = x_i$ and $S_i \circ \gamma = \gamma \circ T_i$ for each $i \in I$, where $T_i(t) = t_{i-1}(1-t) + t_i t$ for $s_i = 0$ and $T_i(t) = t_{i-1} t + t_i(1-t)$ for $s_i = 1$. Moreover, the mapping γ is Hölder continuous and $\gamma(J) = K(\mathbf{S})$.*

Proof. Let $R < 1$ be the maximal dilatation of the mappings in $\mathbf{S}$ and let $r > 0$ be the minimal dilatation of the mappings in $\mathbf{T}$. Put $V = V^{(0)} = \{x_0, \dots, x_n\}$, $W = W^{(0)} = \{0, t_1, \dots, t_{n-1}, 1\}$, $V^{(k)} = \Phi^k(V)$, and $W^{(k)} = \Psi^k(W)$, where Φ and Ψ are the Hutchinson operators of $\mathbf{S}$ and $\mathbf{T}$. The sought mapping γ is defined uniquely on each finite set $W^{(k)}$: for every multi-index $j \in I^k$, put $\gamma(T_j(0)) = S_j(x_0)$ and $\gamma(T_j(1)) = S_j(x_n)$. Correctness of the definition of γ is guaranteed by the coincidence of the signatures of $\mathbf{S}$ and $\mathbf{T}$. Thereby γ is defined on the dense set $W^{(\infty)} = \bigcup_{k=1}^\infty W^{(k)}$ in J. For each $\delta \in (0, r)$ we have a number $k \geq 1$ such that $r^{k+1} \leq \delta \leq r^k$. Since $\mathrm{diam}(T_j(J)) \geq r^k$ for all $j \in I^k$, every pair $a, b \in W^{(\infty)}$ with distance $|a - b| < \delta$ is covered by at most two adjacent (i.e., intersecting) copies of rank k of $K(\mathbf{T})$. By the construction of γ, the images $\gamma(a)$ and $\gamma(b)$ as well lie in the union of at most two intersecting copies of rank k of $K(\mathbf{S})$. Consequently, $\rho(\gamma(a), \gamma(b)) \leq 2R^k \,\mathrm{diam}(V)$. Since $k + 1 \geq \mathrm{Log}(\delta)/\mathrm{Log}(r)$, we have $R^k \leq (1/R)\delta^\alpha$, where $\alpha = \mathrm{Log}(R)/\mathrm{Log}(r)$. Hence, the mapping γ is Hölder continuous with exponent $\alpha = \mathrm{Log}(R)/\mathrm{Log}(r)$ on $W^{(\infty)}$ and thereby extends by continuity to J. Moreover,

$$\gamma(J) = \lim_{k\to\infty} \gamma_k(W^{(k)}) = \lim_{k\to\infty} V^{(k)} = K(\mathbf{S}).$$

The lemma is proven.

The mappings γ of Lemma 1.1 for various collections of points $0 < t_1 < \dots < t_{n-1} < 1$ are called *structure parametrizations* of the attractor of $\mathbf{S}$. A zipper $\mathbf{S}$ is a *Jordan zipper* if and only if one (and hence every) of the structure parametrizations of its attractor establishes a homeomorphism of the interval $J = [0, 1]$ onto $K(\mathbf{S})$.

Theorem 1.2. *Let $\mathbf{S} = \{S_1, \dots, S_n\}$ be a zipper with vertices $\{x_0, \dots, x_n\}$ in a complete metric space $(\mathscr{M}, \rho)$ such that all contractions $S_j : \mathscr{M} \to \mathscr{M}$ are injective. If for arbitrary $i, j \in I$ the set $K_i(\mathbf{S}) \cap K_j(\mathbf{S})$ is empty for $|i - j| > 1$ and is a singleton for $|i - j| = 1$ then every structure parametrization $\gamma : [0, 1] \to K(\mathbf{S})$ of $K(\mathbf{S})$ is a homeomorphism and $K(\mathbf{S})$ is a Jordan arc with endpoints x_0 and x_n.*

Proof. Let $\gamma : J \to K(\mathbf{S})$ be the structure parametrization for $K(\mathbf{S})$ constructed in Lemma 1.1. Consider the system $\mathbf{T} = \{T_1, \dots, T_n\}$ as a zipper with the same signature as that of $\mathbf{S}$, the vertices $\{0, t_1, \dots, t_{n-1}, 1\}$, and the attractor $J = K(\mathbf{T})$.

Note that the following condition is satisfied:

(A) If $x = \gamma(a) = \gamma(b)$ for $a, b \in J$, $a < b$, then either

$(a1)$ $a, b \in K_i(\mathbf{T})$ for some $i \in I$ or

$(a2)$ the relations $a \in K_i(\mathbf{T})$ and $b \in K_{i+1}(\mathbf{T})$ hold for some $i \in I$ and $x = \gamma(t_i)$.

Indeed, assume that $a \in K_i(\mathbf{T})$ and $b \in K_j(\mathbf{T})$. Then $x \in K_i(\mathbf{S}) \cap K_j(\mathbf{S})$ and, by the condition of the theorem, $|i - j| \le 1$. If $i = j$ then $(a1)$ takes place. If $|i - j| = 1$ then it follows from $a < b$ that $j = i + 1$ and we obtain $(a2)$.

Suppose that the mapping γ is not injective. Then there are points $a, b \in J$, $a < b$, with $\gamma(a) = \gamma(b)$. Using (A) and replacing b with t_i in the case $(a2)$, without loss of generality we may assume that the original points a and b lie in the same copy $K_i(\mathbf{T})$ of rank 1 and consequently there is $q = \max\{k : \exists j \in I^k$ such that $a, b \in K_j(\mathbf{T})\} \ge 1$. Suppose that $a, b \in K_{j_0}(\mathbf{T})$ for $j_0 \in I^q$. Then the points $a' = T_{j_0}^{-1}(a)$ and $b' = T_{j_0}^{-1}(b)$ lie in different copies of rank 1 of the attractor $J = K(\mathbf{T})$; moreover, $\gamma(a') = S_{j_0}^{-1} \circ \gamma(a) = S_{j_0}^{-1} \circ \gamma(b) = \gamma(b')$. Consequently, the case $(a2)$ takes place in (A) for the points $a_0 = \min\{a', b'\}$ and $b_0 = \max\{a', b'\}$; i.e., $a_0 \in K_{i_0}(\mathbf{T})$ and $b_0 \in K_{i_0+1}(\mathbf{T})$ for some $i_0 \in I$; moreover, $\gamma(a_0) = \gamma(t_{i_0})$. Then the points $a_0' = T_{i_0}^{-1}(a_0)$ and $b_0' = T_{i_0}^{-1}(t_{i_0}) = 1 - s_{i_0}$ differ and

$$\gamma(a_0') = S_{i_0}^{-1} \circ \gamma(a_0) = S_{i_0}^{-1} \circ \gamma(t_{i_0}) = \gamma(1 - s_{i_0}) = x_{1-s_{i_0}}.$$

We thus arrive at the following condition:

(B) If the parametrization γ is not injective then either

$(b1)$ $\gamma(a) = \gamma(0) = x_0$ for some $a \in (0, 1)$ or

$(b2)$ $\gamma(a) = \gamma(1) = x_n$ for some $a \in (0, 1)$.

Consider the case $(b1)$. Since $\gamma(t_1) = x_1 \ne x_0 = \gamma(0)$, the case $(a2)$ in (A) is impossible for the pair 0, a and therefore $a \in K_1(\mathbf{T})$. Consequently, $q = \max\{k : a \in K_j(\mathbf{T})$, where $j = 11\dots1 \in I^k\} \ge 1$. Let $j_0 = 11\dots1 \in I^q$. Then the points $a_0 = T_{j_0}^{-1}(0) \in \{0, 1\}$ and $a_1 = T_{j_0}^{-1}(a)$ do not lie in the same copy of rank 1 of $J = K(\mathbf{T})$, although $\gamma(a_0) = \gamma(a_1)$. Then the case $(a2)$ must take place in (A) for the points a_0 and a_1. For $a_0 = 0$ we deduce the following contradiction: $x_0 = \gamma(0) = \gamma(t_1) = x_1$. However, $a_0 = 1$ leads to the same contradiction: $x_n = \gamma(1) = \gamma(t_{n-1}) = x_{n-1}$. Therefore, $(b1)$ is impossible. Similarly, we establish that $(b2)$ is impossible. Thus, the assumption that γ is not injective leads to a contradiction. The theorem is proven.

Example 1.3 [1, p. 729]. Considering the complex plane (z), put

$$z_0 = 0, \quad z_1 = 1/2 + i/2\sqrt{3}, \quad z_2 = 1,$$

$$S_1(z) = (1/\sqrt{3})\bar{z}\exp(i\pi/6), \quad S_2(z) = 1 + (1/\sqrt{3})(\bar{z} - 1)\exp(-i\pi/6).$$

The system $\mathbf{S} = \{S_1, S_2\}$ of contraction similarities is a zipper with vertices $\{z_0, z_1, z_2\}$ and signature $(0, 0)$, while its attractor $K(\mathbf{S})$ is the classical *Koch curve*.

Example 1.4. On the complex plane (z), the system $\mathbf{Q}$ of the contraction similarities $Q_1(z) = (1/2)z$, $Q_2(z) = 1/2 + i\sqrt{3}/2 + (1/2)(z - (1 + i\sqrt{3})/2)$, and $Q_3(z) = 1 + (1/2)(z - 1)$ has the attractor $K(\mathbf{Q}) = D$ that is the classical *Sierpiński triangle*. Put

$$z_0 = 0, \quad z_1 = \frac{1 + i\sqrt{3}}{4}, \quad z_2 = \frac{3 + i\sqrt{3}}{4}, \quad z_3 = 1;$$

$$S_1(z) = \frac{1}{2}\bar{z}\exp\left(\frac{i\pi}{3}\right), \quad S_2 = \frac{1}{2}z + \frac{1 + i\sqrt{3}}{4}, \quad S_3(z) = 1 + \frac{1}{2}(\bar{(z)} - 1)\exp\left(-\frac{i\pi}{3}\right).$$

The system $\mathbf{S} = \{S_1, S_2, S_3\}$ is a zipper with vertices $\{z_0, z_1, z_2, z_3\}$ and signature $(0, 0, 0)$ and its attractor $K(\mathbf{S})$ is again the Sierpiński triangle D. This is a consequence of the obvious equalities $S_j(D) = Q_j(D)$ for $j = 1, 2, 3$, the equality $D = S_1(D) \cup S_2(D) \cup S_3(D)$, and uniqueness of the invariant set for $\mathbf{S}$.

EXAMPLE 1.5. In the complex plane ($z = x + iy$), the system $\mathbf{S}$ of the mappings $S_1(x+iy) = xi/2$, $S_2(x+iy) = (x+i)/2$, $S_3(x+iy) = (1+x+i)/2$, and $S_4(x+iy) = 1 + (1-x)i/2$ is a zipper with vertices $\{0, i/2, (1+i)/2, 1+i/2, 1\}$ and signature $(0,0,0,0)$. The attractor $K(\mathbf{S})$ is the union of the segments $[0, i/2]$, $[i/2, 1/2+i/2]$, $[1/2+i/2, 1+i/2]$, and $[1+i/2, 1]$ each of which is a corresponding copy of rank 1. In this example, $K(\mathbf{S})$ is a Jordan arc with endpoints 0 and 1; the copies of rank 1 satisfy the conditions of Theorem 1.2, but no structure parametrization $\gamma : J \to K(\mathbf{S})$ is a homeomorphism (some copies of rank 2 are singletons). Consequently, the condition that the contractions S_i are injections in Theorem 1.2 cannot be omitted.

A zipper $\mathbf{S} = \{S_1, \dots, S_n\}$ is *selfsimilar* if the mappings $S_i : (\mathscr{M}, \rho) \to (\mathscr{M}, \rho)$ are *similarities* with dilatations $r_i \in (0,1)$ (i.e., $\rho(S_i(x), S_i(y)) = r_i \cdot \rho(x,y)$ for all $x, y \in \mathscr{M}$). The unique solution α of the equation $r_1^\alpha + \dots + r_n^\alpha = 1$ is called the *similarity dimension* of $\mathbf{S}$.

2. Jordan zippers with bounded turning. In line with [5, 2.7, p. 100], we introduce the class c-BT ($c \in [1, \infty)$) of metric spaces $(\mathscr{M}, \rho)$ *with bounded turning* defined by the following condition: every pair $a, b \in \mathscr{M}$ can be joined by a continuum $\Gamma \subset \mathscr{M}$ such that $\operatorname{diam}(\Gamma) = \sup_{x,y\in\Gamma} \rho(x,y) \le c\rho(a,b)$.

Lemma 2.1. *Let $\mathbf{S} = \{S_1, \dots, S_n\}$ be a selfsimilar Jordan zipper with vertices $\{x_0, \dots, x_n\}$ and signature $(s_1, \dots, s_n)$ in a complete metric space $(\mathscr{M}, \rho)$. The attractor $K = K(\mathbf{S})$ is an arc with bounded turning ($K \in c$-BT) if and only if the following condition holds at all vertices x_i ($i = 1, \dots, n-1$):*

(U) *there is $c_i \ge 1$ such that the estimate $\operatorname{diam}(\Gamma_{xy}) \le c_i\rho(x,y)$ is valid for every Jordan arc $\Gamma_{xy} \subset K$ with endpoints $x \in K_i(\mathbf{S})$ and $y \in K_{i+1}(\mathbf{S})$.*

PROOF. If $K(\mathbf{S}) \in c$-BT then (U) obviously holds for every $i = 1, \dots, n-1$ with the constant $c_i = c$. To check the converse implication, put $\delta = \min\{\rho(x,y) : x \in K_i(\mathbf{S}), y \in K_j(\mathbf{S}); |i-j| > 1\} > 0$, $d = \operatorname{diam} K$, $c' = \max\{c_1, \dots, c_n\}$, $c'' = d/\delta$, and $c = \max\{c', c''\}$. If points a and b lie in different copies of rank 1 then either $\operatorname{diam}(\Gamma_{ab}) \le c'\rho(a,b)$ (the points lie in adjacent copies of rank 1) or $\operatorname{diam}(\Gamma_{ab}) \le d \le c''\rho(a,b)$ (the points lie in disjoint copies of rank 1). If a and b lie in the same copy of rank 1 then $p = \max\{k : \exists j \in I^k \text{ such that } a, b \in K_j(\mathbf{S})\} \ge 1$. Let $a, b \in K_j(\mathbf{S})$, $j \in I^p$. Then the points $a' = S_j^{-1}(a)$ and $b' = S_j^{-1}(b)$ lie in different copies of rank 1. Since S_j is a similarity with dilatation r_j, from the preceding estimates we obtain the inequality

$$\operatorname{diam}(\Gamma_{ab}) = r_j \operatorname{diam}(\Gamma_{a'b'}) \le r_j \max\{c', c''\}\rho(a', b') = c\rho(a,b).$$

Thus, $K \in c$-BT. The lemma is proven.

Given $\alpha > 0$, we denote by $\mathscr{H}_\alpha$ the conventional α-dimensional Hausdorff measure on the σ-ring of all Borel sets in $\mathbb{R}^d$ and by $\dim_H$, the Hausdorff dimension of a Borel set in $\mathbb{R}^d$.

Theorem 2.2. *If the attractor $K = K(\mathbf{S})$ of a selfsimilar Jordan zipper $\mathbf{S}$ in $\mathbb{R}^d$ has bounded turning and α is the similarity dimension of the system $\mathbf{S}$ then $0 < \mathscr{H}_\alpha(K) < +\infty$ and consequently $\dim_H(K) = \alpha$.*

PROOF. Suppose that $\mathbf{S} = \{S_1, \dots, S_n\}$ is a Jordan zipper with vertices $\{x_0, \dots, x_n\}$, the mappings $S_i : \mathbb{R}^d \to \mathbb{R}^d$ are similarities with dilatations $r_i \in (0,1)$ ($i \in I$), $K \in c$-BT, and $D = \operatorname{diam}(K)$. Assume that $\mathscr{H}_\alpha(K) = 0$. Then (see [6, p. 996]) the identity mapping id is the limit of some sequence uniformly convergent (over compact sets in $\mathbb{R}^d$) of the family $\mathscr{F}$ of all mappings (similarities) of the form $S_i^{-1} \circ S_j$ with multi-indices $i = i_1 \dots i_p \in I^p$ and $j = j_1 \dots j_q \in I^q$, $p, q = 1, 2, \dots$, such that $i_1 \ne j_1$. Put $\varepsilon = D/c$. Then there exist $p \ge 1$, $q \ge 1$, and multi-indices $i \in I^p$ and $j \in I^q$ with $i_1 \ne j_1$ for which

$$\max\bigl\{\bigl|x_0 - S_i^{-1} \circ S_j(x_0)\bigr|, \bigl|x_n - S_i^{-1} \circ S_j(x_n)\bigr|\bigr\} < \varepsilon.$$

Since S_i is a similarity with dilatation $r_i = r_{i_1} r_{i_2} \dots r_{i_p}$, we have

$$\max\{|S_i(x_0) - S_j(x_0)|, |S_i(x_n) - S_j(x_n)|\} < r_i\varepsilon.$$

The points $S_i(x_0), S_i(x_n) \in K_{i_1}$ are the endpoints of the Jordan arc $K_i(\mathbf{S})$ which is a copy of rank p, and the points $S_j(x_0), S_j(x_n) \in K_{j_1}$ are the endpoints of the Jordan arc $K_j(\mathbf{S})$ which is a copy of rank q. It follows from the location of these arcs on K that

$$\max\{\operatorname{diam}(\Gamma_{S_i(x_0),S_j(x_0)}), \operatorname{diam}(\Gamma_{S_i(x_n),S_j(x_n)})\} \geq \operatorname{diam}(K_i) = r_i D.$$

Since $K \in c$-BT, we obtain the inequality

$$r_i D \leq c \max\{|S_i(x_0) - S_j(x_0)|, |S_i(x_n) - S_j(x_n)|\} < c r_i \varepsilon,$$

which leads to the following contradiction: $D < c\varepsilon = D$. The theorem is proven.

Example 2.3. We show that in the plane $(z = x + iy)$ there is a selfsimilar Jordan zipper $\mathbf{S}$ whose attractor $K = K(\mathbf{S})$ is not an arc with bounded turning. Note that, for arbitrary rational numbers d_1, $d_2 \neq 0$ and a natural $n \geq 1$, the number $(d_1 + d_2\sqrt{5})^n$ is irrational. (If $(d_1 + d_2\sqrt{5})^n = A + B\sqrt{5}$ with rational A and B then the Newton binomial formula yields the relation $0 \neq (d_1+d_2\sqrt{5})^n - (d_1-d_2\sqrt{5})^n = 2B\sqrt{5}$ which means that $B \neq 0$.) In particular, for $\tau = (\sqrt{5}-1)/2 \in (0,1)$ the numbers τ^n with $n \geq 1$ cannot be integer powers of 2 and consequently the numbers $\log_2(\tau) \in (-1,0)$ and $\log_2(\tau^2/2) \in (-3,-1)$ are irrational.

Construct the zipper $\mathbf{S} = \{S_1, \ldots, S_4\}$ with vertices $z_0 = -\tau$, $z_1 = 0$, $z_2 = (1+i\sqrt{3})/2$, $z_3 = 1$, and $z_4 = 2+\tau$ and signature $(0,0,0,0)$ which is generated by the similarities S_j with dilatations $r_1 = \tau/(2+2\tau) = \tau^2/2$, $r_2 = r_3 = 1/(2+2\tau) = \tau/2$, and $r_4 = 1/2$ such that S_1 and S_4 preserve the orientation of the plane, whereas S_2 and S_3 change. The triangles $z_0z_1z_2$ and $z_2z_3z_4$ are similar (for $\tau/1 = 1/(1+\tau)$). Denoting the angle at the vertex z_0 by β_0 and denoting the angle of the vertex z_4 by β_1, we observe that $\beta_0+\beta_1 = \pi/3$, the angle $z_0z_2z_1$ equals β_1, and the angle $z_3z_2z_4$ equals β_0. Taking the domain U to be the open triangle with vertices z_0, z_2, and z_4, we see that $S_i(U) \cap S_j(U) = \varnothing$ for $i \neq j$ and consequently the selfsimilar zipper $\mathbf{S}$ satisfies the open set condition (OSC) (see [1, (1), p. 735]) which is equivalent (see [7, Theorem 2.2, p. 114]) to the fact that the Hausdorff dimension $\dim_H(K)$ of the attractor $K = K(\mathbf{S})$ coincides with the similarity dimension α of the system $\mathbf{S}$ and the α-dimensional Hausdorff measure satisfies the inequality $0 < \mathscr{H}^\alpha(K) < +\infty$ (see also [8, Theorem 3, p. 7]). Since $K_j = S_j(K) \subset \overline{S_j(U)}$ for $j = 1, \ldots, 4$, K_1 meets only the copy K_2 at the point z_1; K_4 meets only the copy K_3 at the point z_3, and the copies K_2 and K_3 may meet only at points of the interval $L = [z_2z_5]$, where the point z_5 on the interval $[z_1z_3]$ is such that the angle $z_1z_2z_5$ equals β_1 and the angle $z_5z_2z_3$ equals β_0. Then

$$K_2 \cap L = \{z_2\} \cup \{a_n = S_2(S_4^n(z_2)) : n = 0,1,\ldots\},$$
$$K_3 \cap L = \{z_2\} \cup \{b_n = S_3(S_1^n(z_2)) : n = 0,1,\ldots\},$$
$$\rho_n = |a_n - z_2| = |S_2(S_4^n(z_2)) - S_2(z_4)| = r_2 r_4^n |z_2 - z_4| = \sqrt{2}/2^{n+1},$$
$$\sigma_m = |b_m - z_2| = |S_3(S_1^m(z_0)) - S_3(z_0)| = r_3 r_1^m |z_2 - z_0| = \sqrt{2}(\tau^2/2)^m \tau/2.$$

Therefore, the equality $a_n = b_m$ holds only if $\tau^{2m+1} = 2^{m-n}$ which is impossible for integers m and n. Consequently, the copies K_2 and K_3 meet only at the point z_2 and therefore $\mathbf{S}$ is a Jordan zipper.

Since $\log_2(\tau^2/2)$ is irrational, the set of fractional parts of all numbers of the form $m\log_2(\tau^2/2)$ for $m = 1,2,\ldots$ is dense in the interval $[0,1]$. Therefore, for every $\varepsilon \in (0,1)$, there are naturals m and n for which $|(m\log_2(\tau^2/2)+n) - (-\log_2\tau)| \leq \log_2(1+\varepsilon)$. Since

$$|\log_2(\sigma_m/\rho_n)| = |\log_2[(\tau^2/2)^m \tau 2^n]| = |m\log_2(\tau^2/2) + n + \log_2\tau| \leq \log_2(1+\varepsilon),$$

we have $(1+\varepsilon)^{-1} \leq \sigma_m/\rho_n \leq 1+\varepsilon$. Then the inequality $|a_n - b_m|/(\rho_n+\sigma_m) = |\sigma_m - \rho_n|/(\sigma_m+\rho_n) \leq \varepsilon$ holds for the arc $\gamma \subset K(\mathbf{S})$ with endpoints a_n and b_m. This implies that

$$\operatorname{diam}(\gamma) \geq \max\{\sigma_m, \rho_n\} \geq (\sigma_m+\rho_n)/2 \geq |a_n - b_m|/2\varepsilon.$$

Since ε is arbitrarily small, this means, however, that the arc $K(\mathbf{S})$ does not belong to the class c-BT for any $c < \infty$.

3. Vertices of the first and second types. Observe one geometric property of periodic Jordan arcs on the complex plane $\mathbb{C}$ which is expressed by the following lemma:

Lemma 3.1 (on disjoint periodic arcs). *Suppose that Jordan arcs* Γ_1 *and* Γ_2 *on* $\mathbb{C}$ *with the common beginning* 0 *and endpoints* a_1 *and* a_2 *are disjoint in* $\mathbb{C}\backslash\{0\}$. *If the similarities* $F_j(z) = C_j z$ *with* $|C_j| = \rho_j < 1$ *are such that* $F_j(\Gamma_j) \subset \Gamma_j$ $(j = 1, 2)$ *then*

$$\alpha_1/\operatorname{Log}(\rho_1) = \alpha_2/\operatorname{Log}(\rho_2), \tag{1}$$

where α_j *is the increment of the argument* z *along the arc* $\tau_j \subset \Gamma_j$ *with endpoints* a_j *and* $F_j(a_j)$ *in the direction from* a_j *to* $F_j(a_j)$. *(Note that, with these notations,* $C_j = \rho_j e^{i\alpha_j}$ *and* $\alpha_j = \operatorname{Im}\ (\int_{\tau_j} z^{-1}dz)$*), where the integral along a Jordan arc is well defined* (*see, for example,* [9, Remark 1, p. 78])).

Proof. Proving (1), without loss of generality we may replace the original arcs Γ_j with their subarcs in some closed disk $\overline{B}(0, R_0)$ with the common beginning 0 and endpoints $a'_j \in \partial B(0, R_0)$. Therefore, we can assume that $|a_j| = R_0$ and $\Gamma_j \subset \overline{B}(0, R_0)$.

Consider $z = \exp(w)$ of the plane $(w = p+i\varphi)$ as a universal covering map of $\mathbb{C}\backslash\{0\}$ (see, for example, [10, Chapter 1, § 5]). Moreover, all γ_j of the Jordan arcs Γ_j satisfy the conditions $T_j(\gamma_j) \subset \gamma_j$ for the translation $T_j(w) = w + \operatorname{Log}(\rho_j) + i\alpha_j$ $(j = 1, 2)$ and $\gamma_1 \cap \gamma_2 = \varnothing$. Assume that $k_1 = \alpha_1/\operatorname{Log}(\rho_1) \neq k_2 = \alpha_2/\operatorname{Log}(\rho_2)$. Without loss of generality we consider $k_1 < k_2$. Fix some lift γ_1 of Γ_1 that is a Jordan arc with beginning $W_0 = p_0 + i\varphi_0$ (where $\exp(W_0) = a_1$) in the half-plane $\{p \le p_0 = \operatorname{Log}(R_0)\}$, passing through the point $T_1^n(W_0) = p_0 + n\operatorname{Log}(\rho_1) + i(\varphi_0 + n\alpha_1)$, $n = 0, 1, \ldots$. By periodicity (with period $\operatorname{Log}(\rho_1) + i\alpha_1$), the arc γ_1 lies in some half-strip

$$\Pi = \{w = p + i\varphi : -\infty < p \le p_0, |\varphi - \varphi_0 - k_1(p - p_0)| < M\}.$$

Fix some lift γ_2 of Γ_2 with beginning $W'_0 = p_0 + i\varphi'_0$, $\exp(W'_0) = a_2$, and take a natural N such that $\varphi'_0 + 2\pi N > \varphi_0 + M$. Then the lift $\gamma'_2 = \{w + 2\pi Ni : w \in \gamma_2\}$ of Γ_2 passes through the set

$$\left\{T_2^n(W'_0) = p_0 + n\operatorname{Log}(\rho_2) + i(\varphi'_0 + 2\pi N + n\alpha_2); n = 0, 1, \ldots\right\}$$

of points of the straight line with slope k_2. The point $W'_0 + 2\pi Ni$ lies above the half-strip Π and, since $k_2 > k_1$, the points $T_2^n(W'_0)$ lie below Π for all sufficiently large n. Consequently, the arc γ'_2 intersects Π and hence γ_1. The lifts of Γ_1 cannot meet the lifts of Γ_2. This contradiction shows that $k_1 = k_2$. The lemma is proven.

Given a selfsimilar Jordan zipper $\mathbf{S} = \{S_1, \ldots, S_n\}$ with vertices $x_0, \ldots, x_n$ and signature $(s_1, \ldots, s_n)$ on the plane $\mathbb{R}^2$, we say that a vertex x_p $(p \in \{1, \ldots, n-1\})$ is a *vertex of the first type* if $s_p = s_{p+1}$ and $s_1 = s_n = 0$; otherwise x_p is a *vertex of the second type.*

Lemma 3.2. *Let* x_p, $p = 1, \ldots, n-1$, *be a vertex of a selfsimilar Jordan zipper* $\mathbf{S} = \{S_1, \ldots, S_n\}$ *with signature* $(s_1, \ldots, s_n)$. *Suppose that*

(i) x_p *is a vertex of the second type or*

(ii) x_p *is a vertex of the first type and the number*

$$\frac{\operatorname{Log}|x_1 - x_0| - \operatorname{Log}|x_n - x_0|}{\operatorname{Log}|x_n - x_{n-1}| - \operatorname{Log}|x_n - x_0|} \tag{2}$$

is rational.

Then the condition (U) *of Lemma* 2.1 *holds for the vertex* x_p.

Proof. Let $x_q = S_p^{-1}(x_p)$ and $x_{q'} = S_{p+1}^{-1}(x_p)$, where $q, q' \in \{0, n\}$. Let $w, w' \in I^\infty$ be the index coordinates of x_q and $x_{q'}$; i.e., $x_q = \pi(w)$ and $x_{q'} = \pi(w')$, where $\pi : I^\infty \to K = K(\mathbf{S})$ is the index parametrization of the attractor K. The following relations are immediate:

For $s_p = s_{p+1}$ the equality $q' = n - q$ holds and

(A1) $w = 111\ldots$ and $w' = nnn\ldots$ for $q = 0$, $s_1 = 0$, and $s_n = 0$;

(A2) $w = 111\ldots$ and $w' = n111\ldots$ for $q = 0$, $s_1 = 0$, and $s_n = 1$;

(A3) $w = 1nnn\ldots$ and $w' = nnn\ldots$ for $q = 0$, $s_1 = 1$, and $s_n = 0$;

(A4) $w = 1n1n\dots1n\dots$ and $w' = n1n1\dots n1\dots$ for $q = 0$, $s_1 = 1$, and $s_n = 1$;
(A5) $w = nnn\dots$ and $w' = 111\dots$ for $q = n$, $s_1 = 0$, and $s_n = 0$;
(A6) $w = n111\dots$ and $w' = 111\dots$ for $q = n$, $s_1 = 0$, and $s_n = 1$;
(A7) $w = nnn\dots$ and $w' = 1nnn\dots$ for $q = n$, $s_1 = 1$, and $s_n = 0$;
(A8) $w = n1n1\dots n1\dots$ and $w' = 1n1n\dots1n\dots$ for $q = n$, $s_1 = 1$, and $s_n = 1$.

For $s_p \neq s_{p+1}$ the equality $q' = q$ holds and
(B1) $w = w' = 111\dots$ for $q = 0$, $s_1 = 0$, and every s_n;
(B2) $w = w' = 1nnn\dots$ for $q = 0$, $s_1 = 1$, and $s_n = 0$;
(B3) $w = w' = 1n1n1n\dots$ for $q = 0$, $s_1 = 1$, and $s_n = 1$;
(B4) $w = w' = nnn\dots$ for $q = n$, $s_n = 0$, and every s_1;
(B5) $w = w' = n111\dots$ for $q = n$, $s_n = 1$, and $s_1 = 0$;
(B6) $w = w' = n1n1n1\dots$ for $q = n$, $s_n = 1$, and $s_1 = 1$.

Note that in each case we can rewrite the multi-indices w and w' as $w = abbb\dots$ and $w' = a'b'b'b'\dots$, where (A2) $a = 1$, $a' = n$, and $b = b' = 11$; (A3) $a = 1$, $a' = n$, and $b = b' = nn$; (A4) $a = 1, a' = n$, $b = n1n1$, and $b' = 1n1n$; (A6) $a = n$, $a' = 1$, and $b = b' = 11$; (A7) $a = n$, $a' = 1$, and $b = b' = nn$; (A8) $a = n$, $a' = 1$, $b = 1n1n$, and $b' = n1n1$; (B1) $a = a' = 1$ and $b = b' = 11$; (B2) $a = a' = 1$ and $b = b' = nn$; (B3) $a = a' = 1$ and $b = b' = n1n1$; (B4) $a = a' = n$ and $b = b' = nn$; (B5) $a = a' = n$ and $b = b' = 11$; (B6) $a = a' = n$ and $b = b' = 1n1n$. In the cases (A1) and (A5), by rationality of (2), there are coprime naturals P and Q for which the dilatations r_1 and r_n of the similarities S_1 and S_n are connected by the equality $r_1^Q = r_n^P$. Put (A1) $a = 1$, $a' = n$, $b = 1\dots1 \in I^{2Q}$, and $b' = n\dots n \in I^{2P}$; (A5) $a = n$, $a' = 1$, $b = n\dots n \in I^{2P}$, and $b' = 1\dots1 \in I^{2Q}$. Observe that the similarities S_b and S'_b in all cases preserve orientation, have the respective fixed points $S_a^{-1}(x_q)$ and $S_{a'}^{-1}(x_{q'})$, and their dilatations r_b and $r_{b'}$ coincide, $r_b = r_{b'}$.

The Jordan arc $\tau = S_{pa}(K) \subset S_p(K) = K_p$ has the endpoint x_p, and the orientation-preserving similarity $A = S_p \circ S_a \circ S_b \circ S_a^{-1} \circ S_p^{-1}$ with dilatation r_b and fixed point x_p is such that $A(\tau) = (A \circ S_p \circ S_a)(K) = S_{pa}(S_b(K)) \subset S_{pa}(K) = \tau$. By analogy, the Jordan arc $\tau' = S_{(p+1)a'}(K) \subset S_{p+1}(K) = K_{p+1}$ has the endpoint x_p, and the orientation-preserving similarity $A' = S_{p+1} \circ S_{a'} \circ S_{b'} \circ S_{a'}^{-1} \circ S_{p+1}^{-1}$ with dilatation $r_{b'}$ and fixed point x_p is such that $A'(\tau') \subset \tau'$. Since the zipper is Jordan, $\tau \cap \tau' = \{x_p\}$ and therefore Lemma 3.1 applies to the arcs τ and τ' and the corresponding similarities A and A'. Since $A = r_b e^{i\alpha}$ and $A' = r_b e^{i\alpha'}$, from equality (1) of Lemma 3.1 we infer the equality $\alpha = \alpha' (\operatorname{mod} 2\pi)$ and the coincidence of these similarities, $A' = A$.

Suppose that $D = \operatorname{diam}(K)$, $\delta = \inf\{|x - y| : (x \in K_p \backslash A(\tau),\ y \in K_{p+1})$ or $(x \in K_p,\ y \in K_{p+1} \backslash A'(\tau'))\}$, and r is the maximal dilatation of the similarities $S_1, \dots, S_n$. For an arbitrary pair $x \in K_p$, $y \in K_{p+1}$ of points at least one of which differs from x_p, find a minimal number $k \geq 1$ such that $x \notin A^k(\tau)$ or $y \notin A^k(\tau')$. If $k = 1$ then $x \in K_p \backslash A(\tau)$ or $y \in K_{p+1} \backslash A(\tau')$, and in this case $|x - y| \geq \delta$. We have the following estimate for the diameter of the arc $\Gamma_{xy} \subset K_p \cap K_{p+1}$:

$$\operatorname{diam}(\Gamma_{xy}) \leq \operatorname{diam}(K_p) + \operatorname{diam}(K_{p+1}) \leq 2rD \leq 2rD\delta^{-1}|x - y|. \tag{3}$$

If $k > 1$ then $x \in A^{k-1}(\tau)$ and $y \in A^{k-1}(\tau')$; moreover, $x \notin A^k(\tau)$ or $y \notin A^k(\tau')$. Then $\tilde{x} = A^{1-k}(x) \in \tau$ and $\tilde{y} = A^{1-k}(y) \in \tau'$; moreover, $\tilde{x} \notin A(\tau)$ or $\tilde{y} \notin A(\tau')$. Recalling that $\Gamma_{xy} = A^{k-1}(\Gamma_{\tilde{x}\tilde{y}})$, we obtain the equalities

$$\operatorname{diam}(\Gamma_{xy}) = r_b^{k-1} \operatorname{diam}(\Gamma_{\tilde{x}\tilde{y}}), \quad |\tilde{x} - \tilde{y}| = |A^{1-k}(x) - A^{1-k}(y)| = r_b^{1-k}|x - y|.$$

Using them in estimate (3) for the pair $\tilde{x}$, $\tilde{y}$, we arrive at the relation

$$\operatorname{diam}(\Gamma_{xy}) \leq 2rD\delta^{-1}|x - y|.$$

We have thus demonstrated that the vertex x_p satisfies the condition (U) of Lemma 2.1 with the constant $c_p = 2rD\delta^{-1}$. The lemma is proven.

Example 2.3 demonstrates that the rationality of (2) is essential in the case of vertices of the first type.

4. Basic theorems on selfsimilar Jordan zippers. The theorems of this section are immediate from applying Lemmas 2.1 and 3.2 to some particular cases important in applications.

Theorem 4.1. *Let* $\mathbf{S} = \{S_1, \dots, S_n\}$ *be a selfsimilar Jordan zipper with vertices* $z_0, \dots, z_n$ *on the plane* $\mathbb{R}^2$. *If the number*

$$\frac{\operatorname{Log}|z_1 - z_0| - \operatorname{Log}|z_n - z_0|}{\operatorname{Log}|z_n - z_{n-1}| - \operatorname{Log}|z_n - z_0|}$$

is rational then the attractor $K = K(\mathbf{S})$ *has bounded turning.*

Theorem 4.2. *If a selfsimilar Jordan zipper* $\mathbf{S}$ *on the plane* $\mathbb{R}^2$ *has no vertices of the first type then its attractor* $K = K(\mathbf{S})$ *is an arc with bounded turning.*

The following two assertions are particular instances of Theorem 4.2:

Theorem 4.3. *If a selfsimilar Jordan zipper* $\mathbf{S} = \{S_1, \dots, S_n\}$ *on the plane has signature* $(s_1, \dots, s_n)$ *such that* $s_1 + s_n \geq 1$ *then its attractor is an arc with bounded turning.*

Theorem 4.4. *If a selfsimilar Jordan zipper* $\mathbf{S} = \{S_1, \dots, S_n\}$ *on the plane* $\mathbb{R}^2$ *has an alternating signature; i.e.,* $s_p \neq s_{p+1}$ *for all* $p = 1, \dots, n-1$, *then its attractor is an arc with bounded turning.*

Remark 1. In the scheme behind the construction of a zipper with alternating signature, we could see a two-dimensional generalization of the *inverse refrain* scheme which was suggested by P. Tukia in [11, p. 154] as a modification of Salem's construction and which was used by P. Tukia for constructing special quasisymmetric homeomorphisms of an interval onto itself.

Remark 2. Theorems 4.1–4.4 do not exhaust all cases in which the attractor of a Jordan zipper on the plane has bounded turning, since this property can result from a suitable disposition of vertices (for example, when all vertices lie on one straight line and the attractor is an interval).

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